

Function Representation

FRep

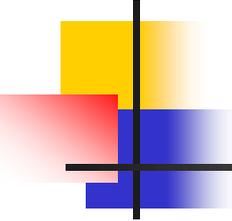
<http://hm.softalliance.net/>

Alexander Pasko, Evgenii Maltsev



Motivation

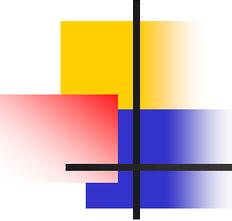
- Links of “implicit” with other models
- Single real-valued function
- Solid sweeps
- Blends in CAGD
- Deformations
- Collision detection
- Metamorphosis



Motivation:

Links With Other Models

- **CSG**: algebraic (quadric) primitives of CSG; algebraic patches in constructive shells; ray-tracing algorithms.
- **BRep**: algebraic patches; polygonization.
- **Sweeping**: generalized cylinders with skeletal curves.
- **Voxels**: polygonization algorithms.



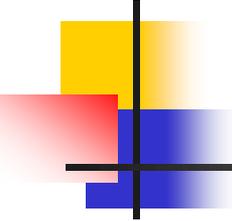
Motivation:

Single Real-valued Function

The idea of representation of an **entire** complex object by a single real function:

- applied effectively in **skeletal implicits**;
- **min/max** operations for CSG proposed by Ricci [1973] have not found wide acceptance (C^1 discontinuity as a reason);
- existence of **R-functions** proposed by Rvachev [1963], surveyed by Shapiro [1988], and applied in multidimensional geometric modeling by Pasko [Ph.D. thesis, 1988].

More deep connection with *CSG* is possible.

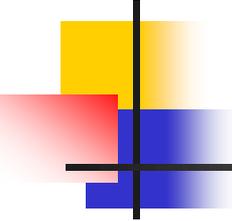


Motivation:

Solid Sweeps

- Theoretical possibility to derive an implicit description of a **surface swept by a moving solid** [Wang 1984].
- Symbolic computations required to yield a formula for the implicit form.

More deep connection with *sweeping* is possible.



Motivation:

Blends in CAGD

- Attention is paid in Computer Aided Geometric Design (CAGD) to implicit surfaces because of their closure under some important operations: **offsetting** and **blending** [Hoffman 1993].

Motivation:

Deformations and Collisions

Deformations available for “implicit”:

- Twist, bend, taper [Barr]
- Free vibrations
[Sclaroff and Pentland 1991].

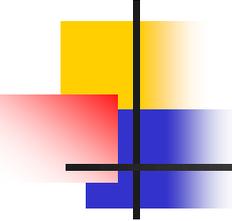
Collision detection algorithms for implicit surfaces [Gascuel 1993]

Motivation:

Voxels as Discrete Fields

- Similar **polygonization** algorithms stress common nature of implicit and voxel models.
- **Metamorphosis** of skeletal implicits [Wyvill 1991] and scheduled Fourier volume morphing [Hughes 1992].

More deep connection with *voxel models* is possible.

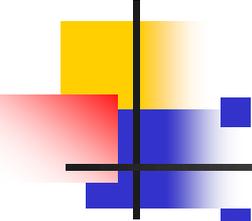


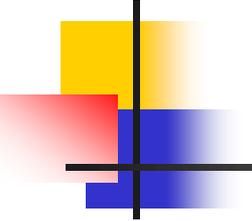
Motivation:

Survey Conclusion

- Representations by real-valued functions are widely used in geometric modeling and computer graphics in several forms.
- These models are not closely related to each other.
- These models are not closely related to such well-known representations as CSG, B-rep, sweeping, and spatial partitioning (voxel models).

Motivation: Survey Conclusion

- 
- This obviously retards further R&D.
 - A uniform *function representation* is needed to fill these gaps.
 - This representation has to:
 - unify all functionally based approaches;
 - be convertible from other representations;
 - be dimension independent;
 - have as rich as possible a system of operations and relations.



FRep: Geometric Concepts

Geometric concepts of a functionally based modeling environment:

$$(\mathbf{M}, \Phi, \mathbf{W})$$

where

M is a set of *geometric objects*,

Φ is a set of *geometric operations*,

W is a set of *relations* on the set of objects.

Mathematically this triple is a sort of
algebraic system.

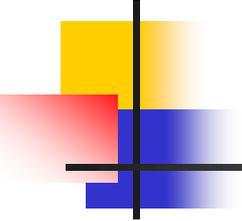


Objects

Closed subsets of n -dimensional Euclidean space E^n with the definition:

$$F(x_1, x_2, \dots, x_n) \geq 0$$

where F is a real-valued continuous function defined on E^n .



⇒ Classification of points in E^n space:

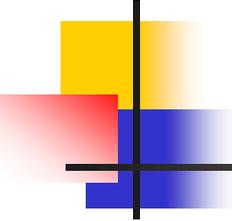
$F(\mathbf{X}) > 0$ - for points inside the object;

$F(\mathbf{X}) = 0$ - for points on the object's
boundary;

$F(\mathbf{X}) < 0$ - for points outside the object.

Here, $\mathbf{X} = (x_1, x_2, \dots, x_n)$ is a point in E^n .

Objects

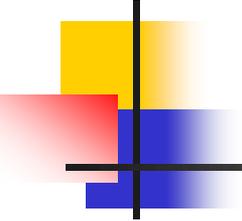


⇒ In 3D space, the boundary of such an object is a so-called “*implicit*” *surface*.

⇒ The function can be defined by:

- 1) analytical expression;
- 2) function evaluation algorithm;
- 3) tabulated values and an appropriate interpolation procedure.

⇒ The major requirement to the function is to have at least **C^0 continuity**

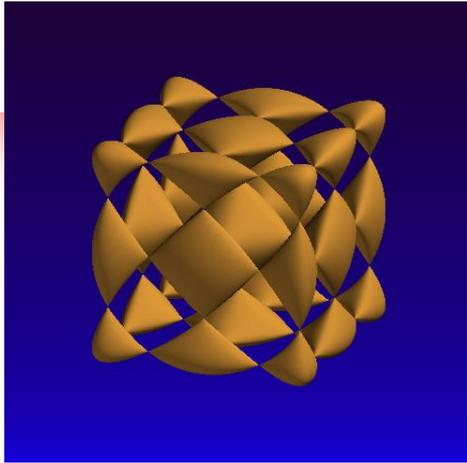
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- ⇒ The inequality with the **explicit function of n variables**, but **not the implicit function** of $n-1$ variables $f(x_1, x_2, \dots, x_n) = 0$.
 - ⇒ **Multidimensional** formulation.
 $n = 4$: space-time modeling.
 - ⇒ ***Primitives*** and ***complex objects***.
A complex geometric object is a result of operations on primitives.
 - ⇒ **Extensibility**



Types of Primitives

- Algebraic surfaces
- Skeletal objects (soft, blobby, etc.)
- Convolution surfaces
- Radial-basis functions (volume splines)
- Trivariate B-splines
- Voxel array + interpolation
- Procedural

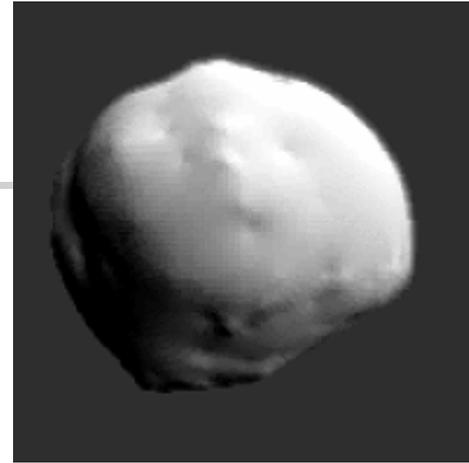
Primitives



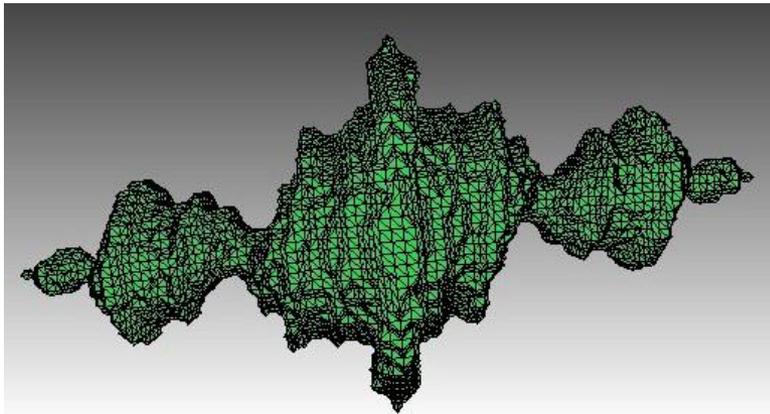
Algebraic



Voxel array



Radial-basis RBF



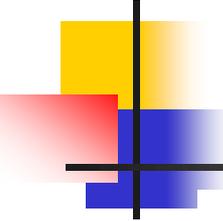
Procedural fractal



Trivariate B-spline

Complex Object: Single Function





Operations

$$\Phi_i: M^1 + M^2 + \dots + M^n \rightarrow M$$

⇒ Two main classes:

1) **unary** ($n=1$) operations

$$G_2 = \Phi_i(G_1)$$

$$f_2 = \Phi(f_1(\mathbf{X})) = 0$$

2) **binary** ($n=2$) operations

$$G_3 = \Phi_i(G_1, G_2)$$

$$f_3 = \Phi(f_1(\mathbf{X}), f_2(\mathbf{X})) = 0$$



Relations

A binary relation is a subset of the set $M^2 = M \times M$. It can be defined as

$$S_i: M \times M \rightarrow I$$

The examples of binary relations are inclusion, point membership, interference or collision.



Point Membership Relation

$$S_3(P, G_1) = \begin{cases} 0, & \text{if } f_1(\mathbf{X}) < 0 & \text{for } P \notin G_1 \\ 1, & \text{if } f_1(\mathbf{X}) = 0 & \text{for } P \in bG_1 \\ 2, & \text{if } f_1(\mathbf{X}) > 0 & \text{for } P \in iG_1 \end{cases}$$

P is a point

iG₁ is the interior of *G₁*

bG₁ is the boundary of *G₁*.



Unary Operations

- Bijective space mapping (deformations)
- Offsetting
- Projection
- Sweeping

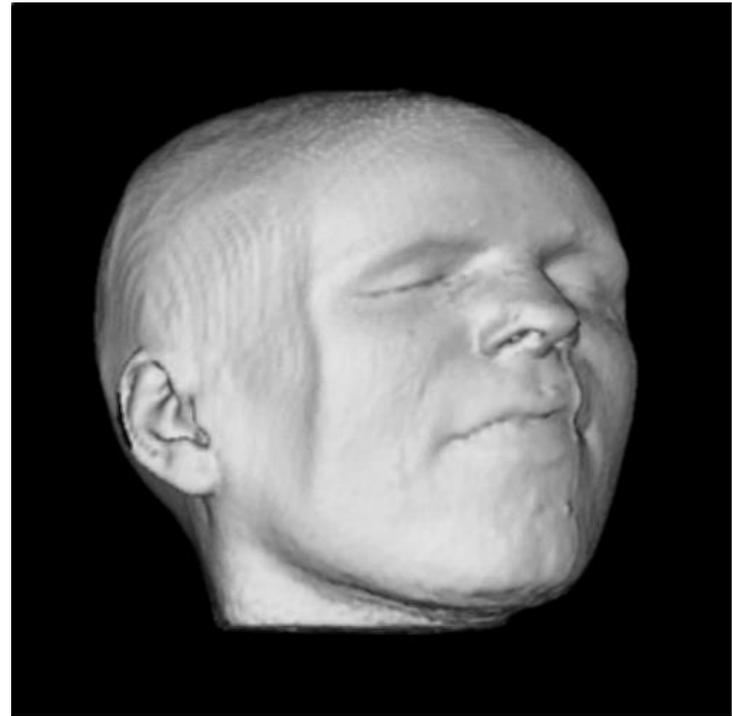
Deformations: Twisting



Feature-based mapping

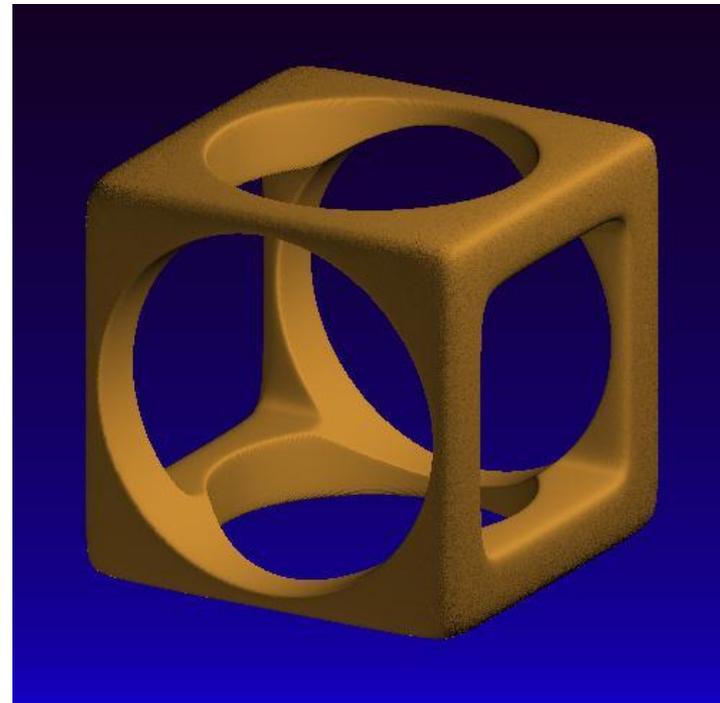
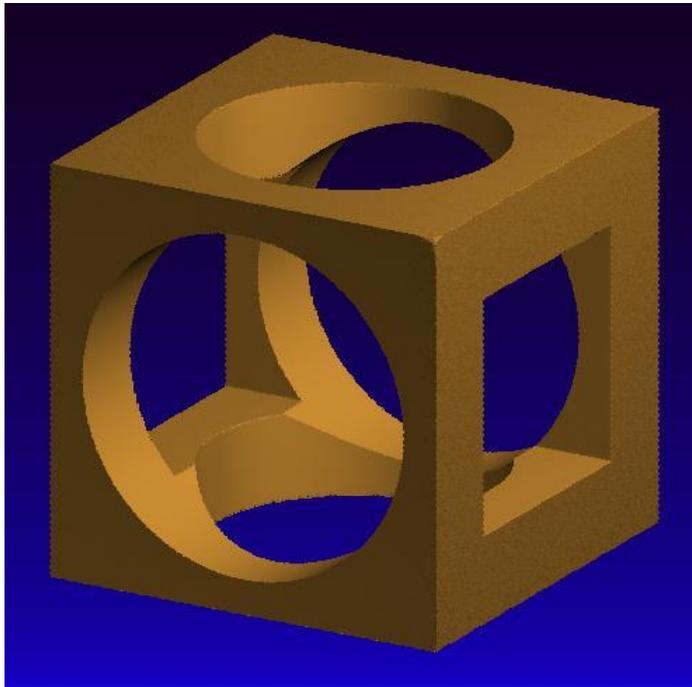


Feature-based mapping

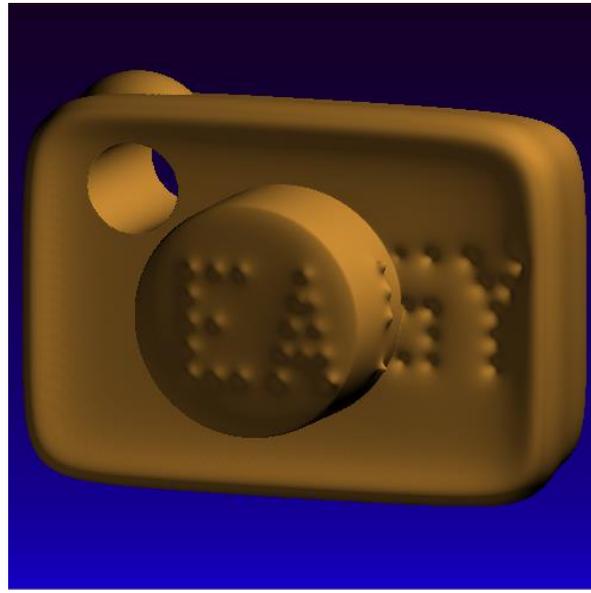
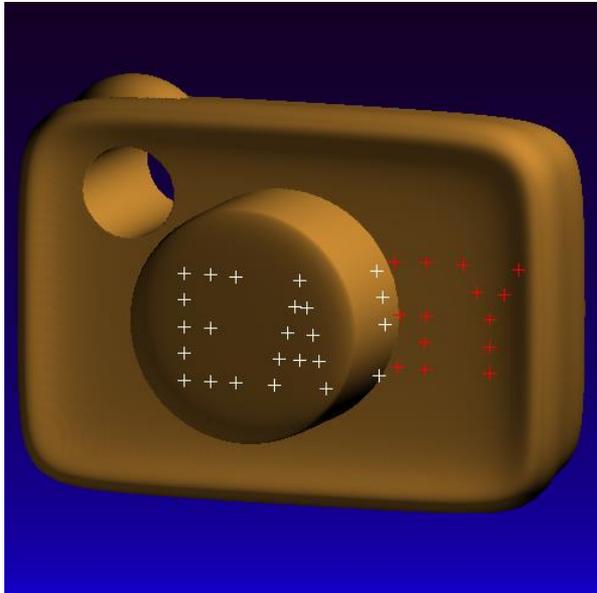


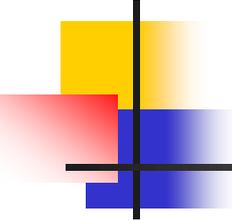


Unary Operations: Offsetting

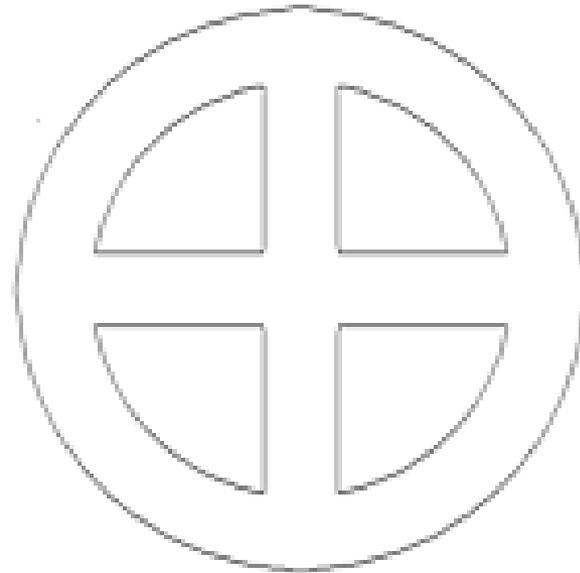
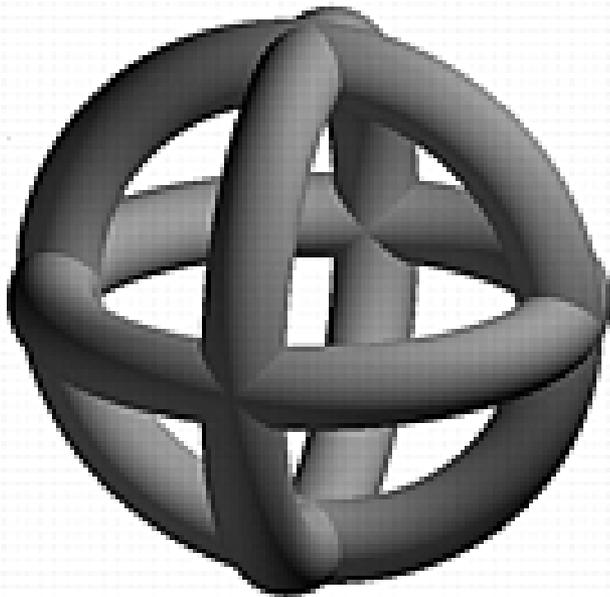


Feature-based offsetting

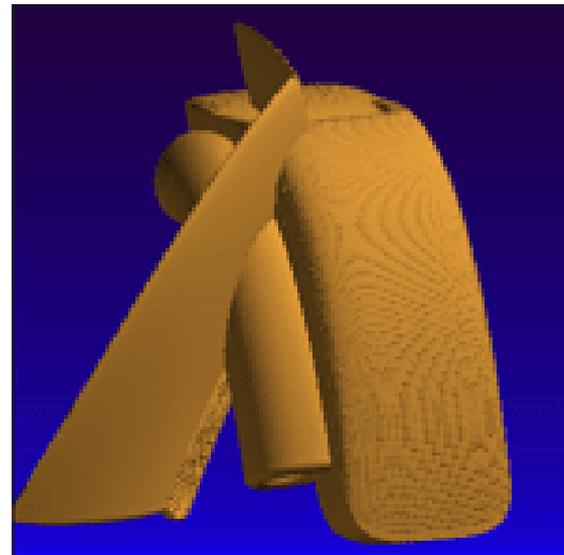
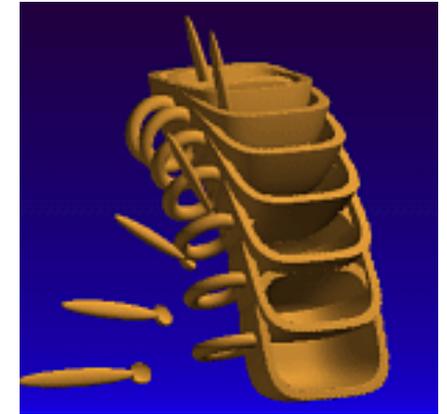
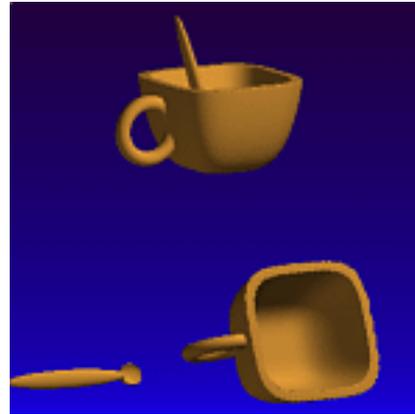
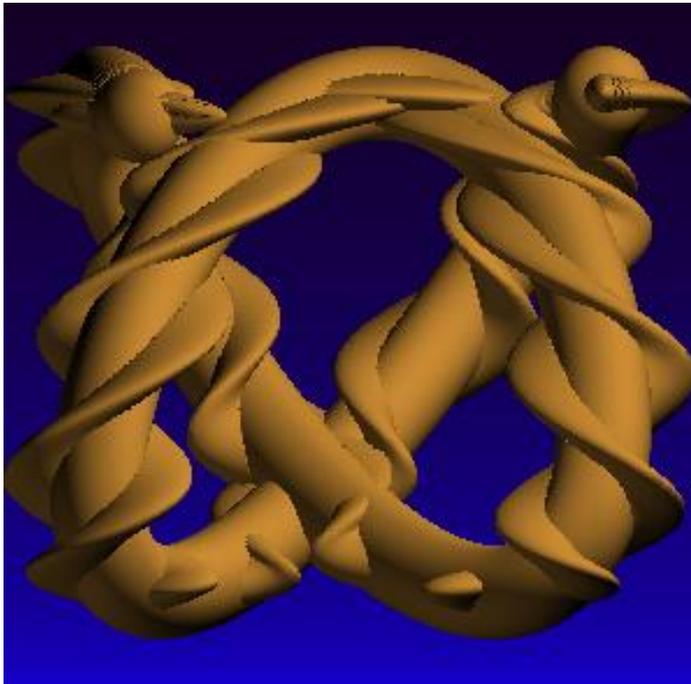




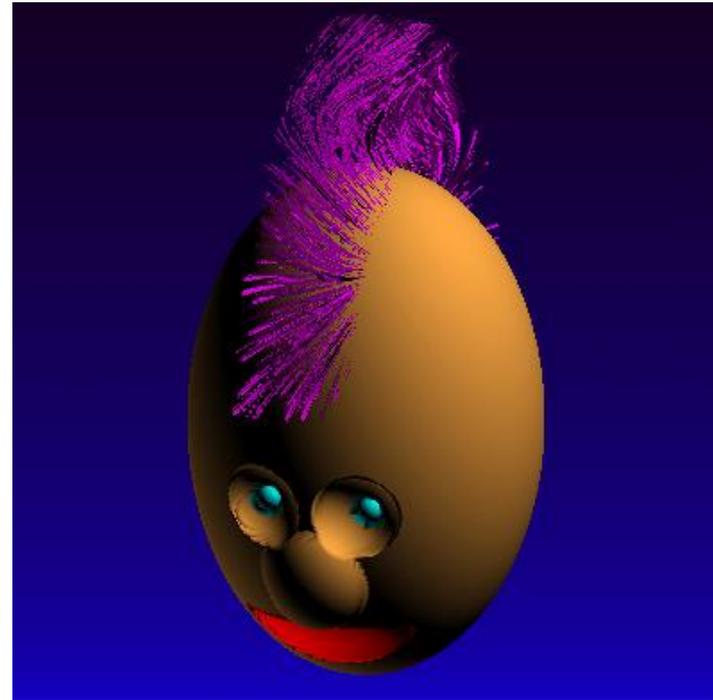
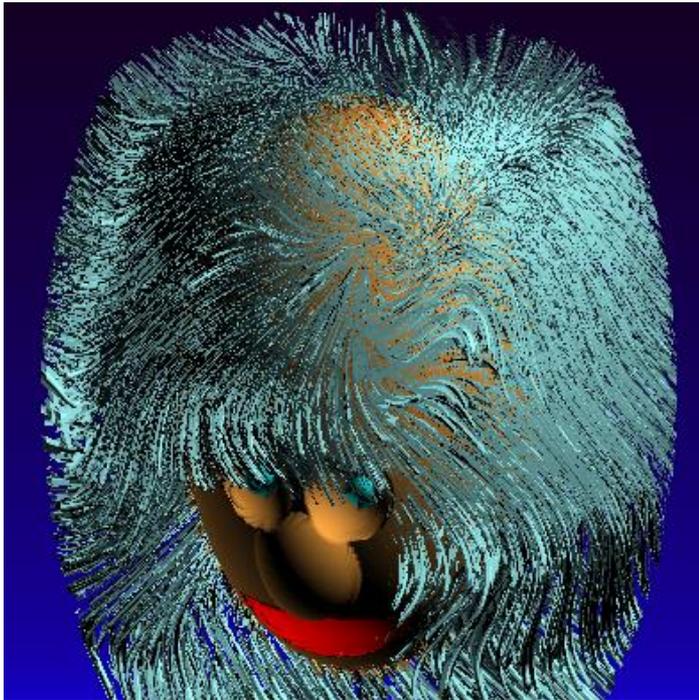
Unary Operations: Projection



Unary Operations: Sweeping



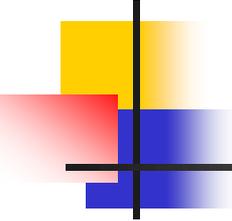
Unary Operations: Hypertexturing





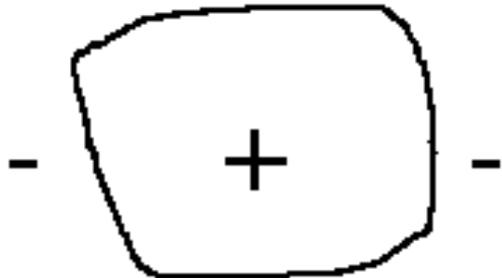
Binary Operations

- Set-theoretic operations:
 - Intersection
 - Union
 - Difference
- Blending
- Metamorphosis

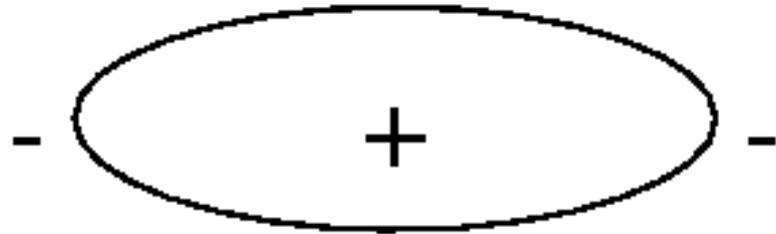


Set-theoretic Operations

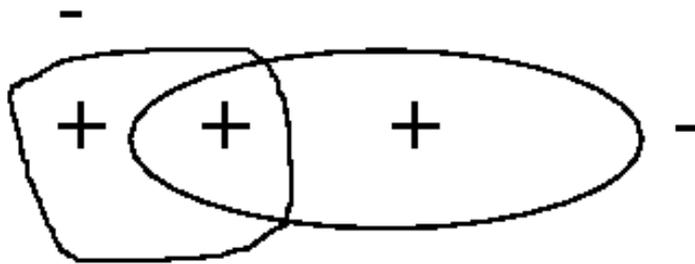
$$G_1: f_1(\mathbf{X}) \geq 0$$



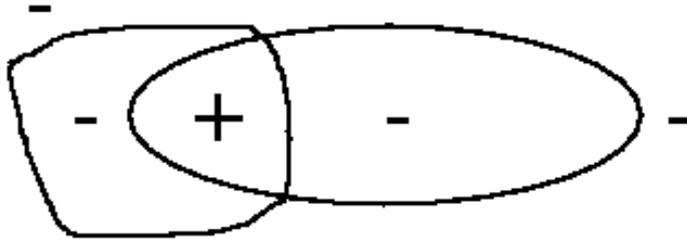
$$G_2: f_2(\mathbf{X}) \geq 0$$



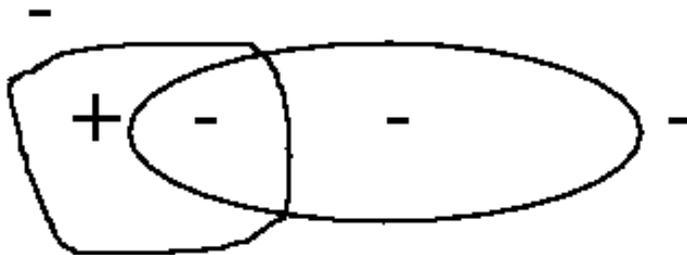
Set-theoretic Operations



$$G_3 = G_1 \cup G_2$$

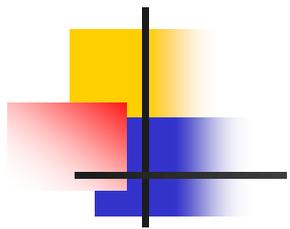


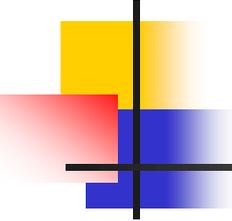
$$G_3 = G_1 \cap G_2$$



$$G_3 = G_1 \setminus G_2$$

$$f_3 = \Psi(f_1(\mathbf{X}), f_2(\mathbf{X})) \quad ?$$





Types of R-functions

Union $G_3 = G_1 \cup G_2 \rightarrow f_3 = f_1 \mid f_2$

Intersection $G_3 = G_1 \cap G_2 \rightarrow f_3 = f_1 \& f_2$

Subtraction $G_3 = G_1 \setminus G_2 \rightarrow f_3 = f_1 \setminus f_2$

$\mid, \&, \setminus$ are signs of R-functions.

Types of R-functions

Rvachev [1972]

$$f_1 \uparrow f_2 = \frac{1}{1 + \alpha} (f_1 + f_2 + \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2})$$

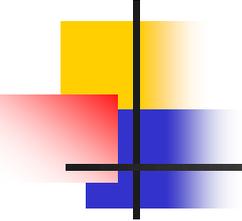
$$f_1 \& f_2 = \frac{1}{1 + \alpha} (f_1 + f_2 - \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2})$$

$$f_1 \setminus f_2 = f_1 \& (-f_2)$$

$$-1 < \alpha(f_1, f_2) \leq 1,$$

$$\alpha(f_1, f_2) = \alpha(f_2, f_1) = \alpha(-f_1, f_2) = \alpha(f_1, -f_2)$$

Types of R-functions



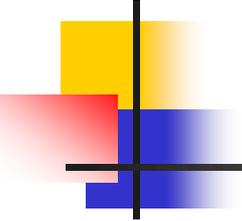
If $\alpha=0$, the above functions take the most useful in practice form:

$$f_1 \vee f_2 = f_1 + f_2 + \sqrt{f_1^2 + f_2^2}$$

$$f_1 \& f_2 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}$$

These functions have C^1 discontinuity only in points where both arguments are equal to zero.

Types of R-functions

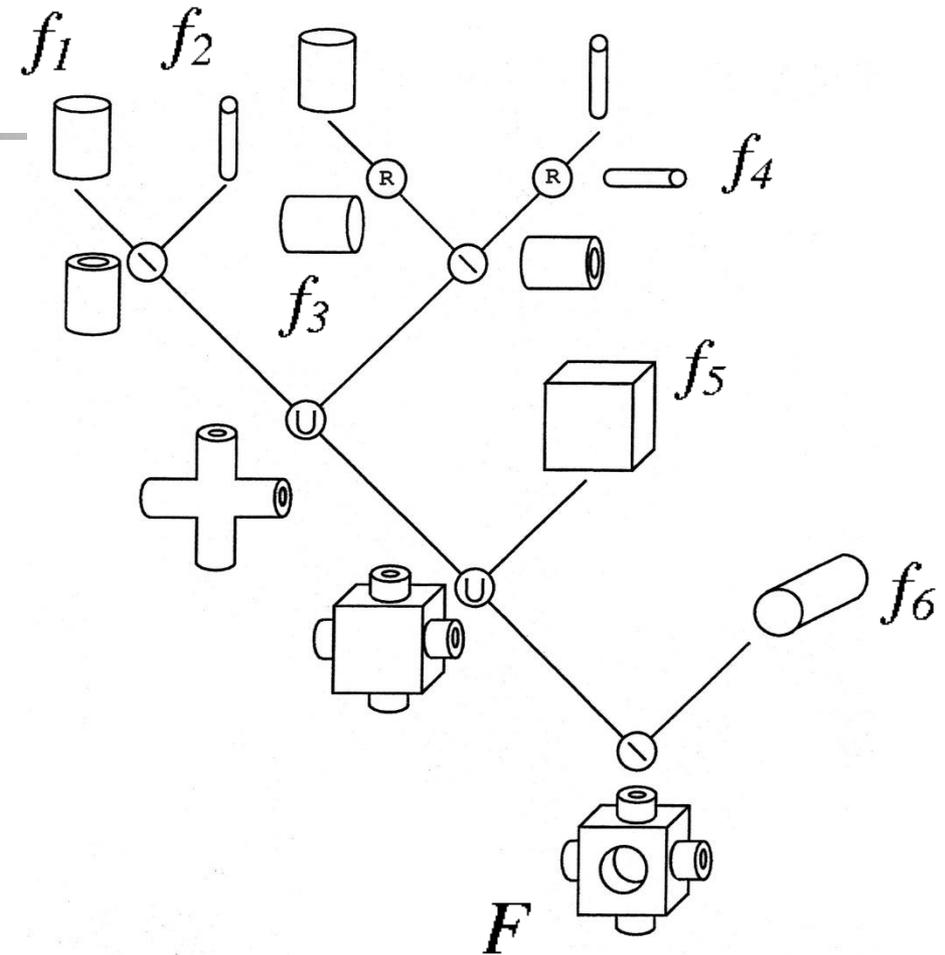
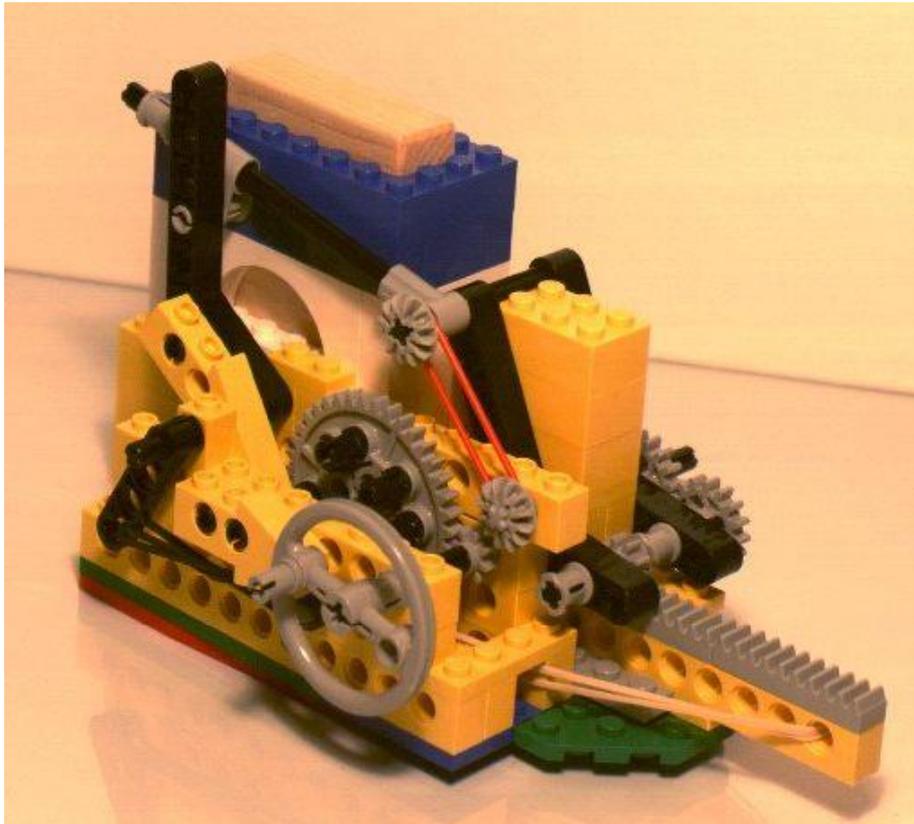


If C^m continuity is to be provided, one may use another set of R-functions:

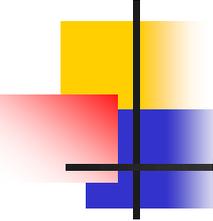
$$f_1 \uparrow f_2 = \left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2} \right) (f_1^2 + f_2^2)^{\frac{m}{2}}$$

$$f_1 \& f_2 = \left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2} \right) (f_1^2 + f_2^2)^{\frac{m}{2}}$$

FRep is constructive as Lego

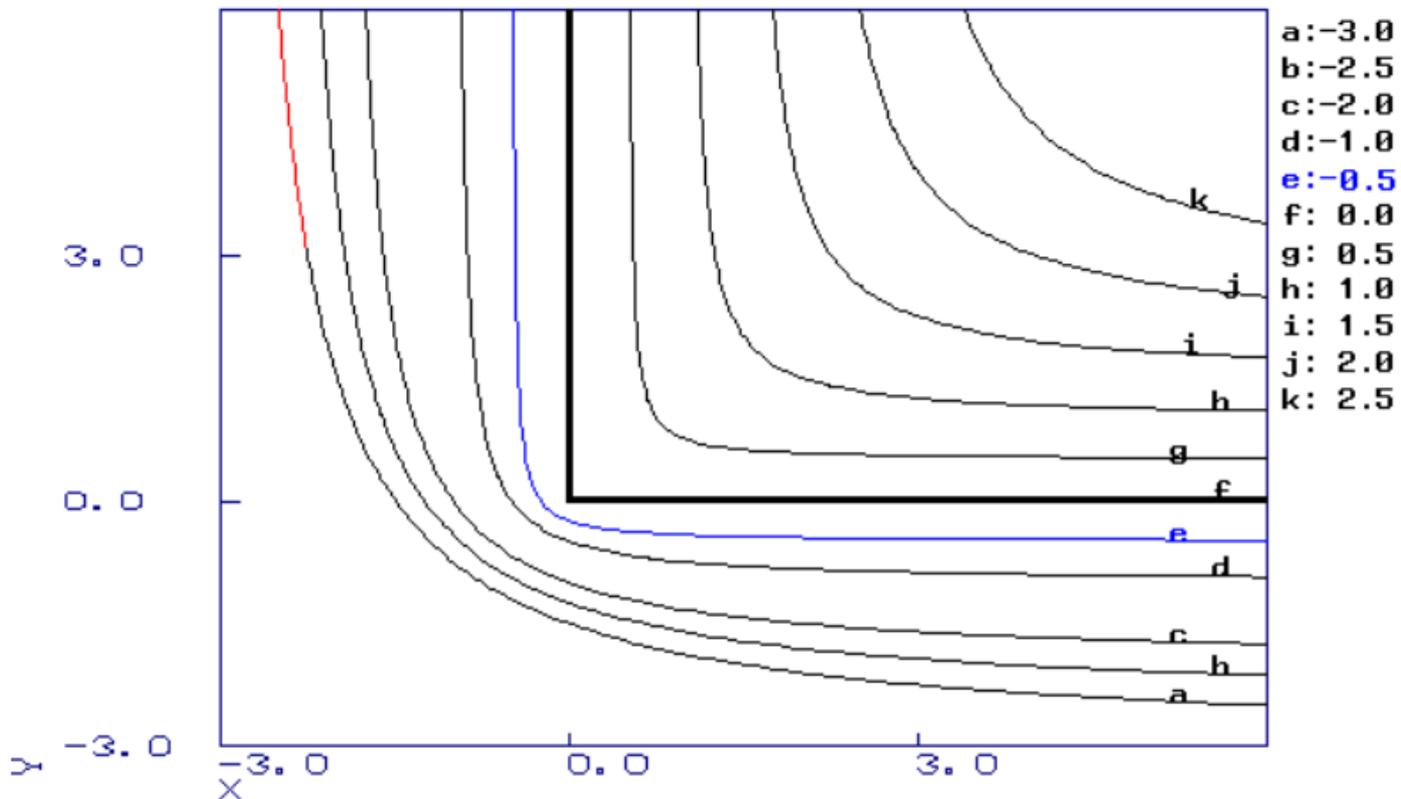


$$F = (((f_1 \cap f_2) \cup (f_3 \cap f_4)) \cup f_5) \cap f_6$$



Contours of R-functions

$$f = x + y - \sqrt{x^2 + y^2}$$

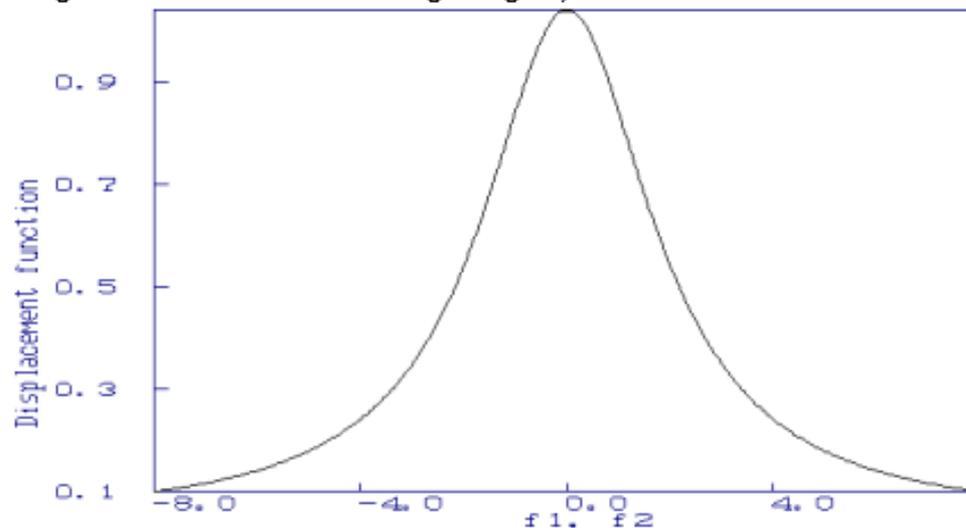


Blending

$$F(f_1, f_2) = R(f_1, f_2) + d(f_1, f_2)$$

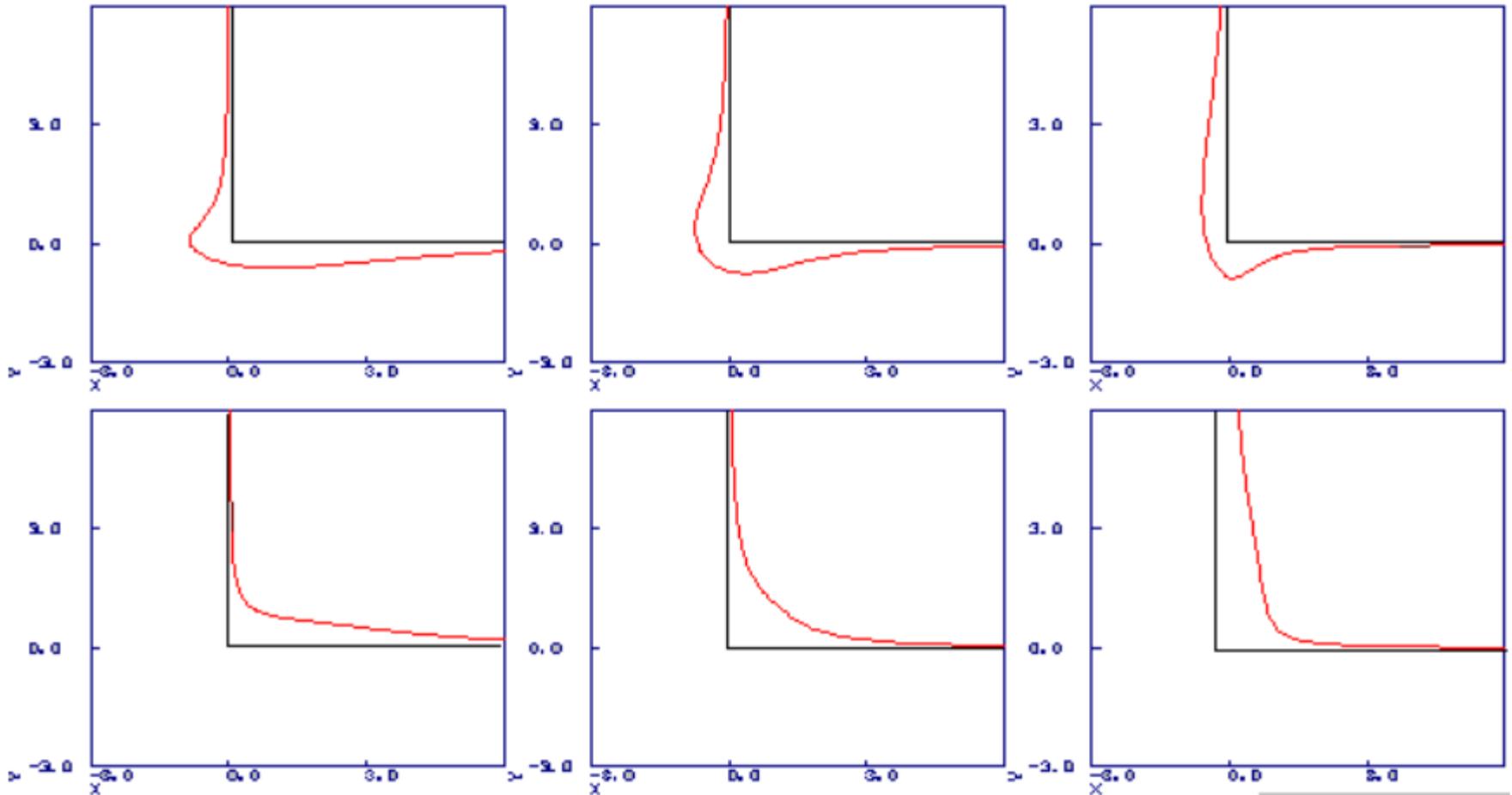
$$d(f_1, f_2) = a_0 / (1 + (f_1/a_1)^2 + (f_2/a_2)^2)$$

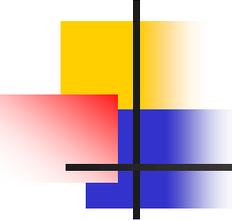
The section $f_1=0$ for $d(f_1, f_2)$:



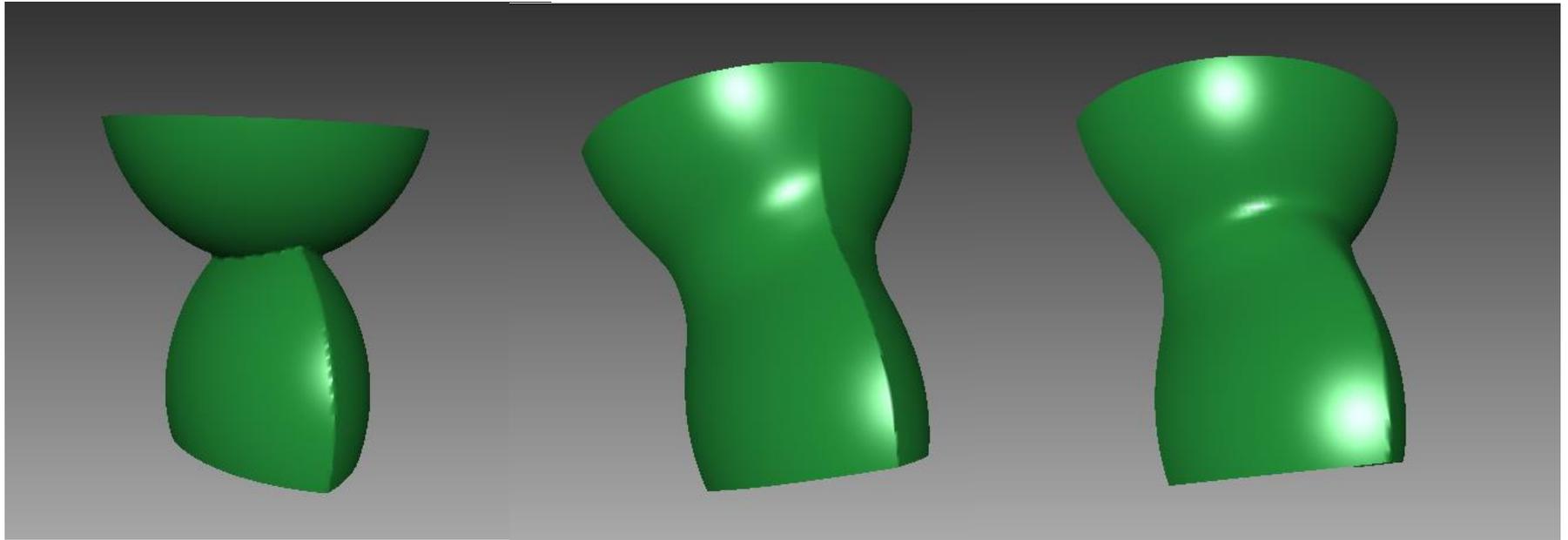
Blending Intersection

$$F(f_1, f_2) = f_1 + f_2 - \sqrt{f_1^2 + f_2^2} + a_0 / (1 + (f_1/a_1)^2 + (f_2/a_2)^2)$$





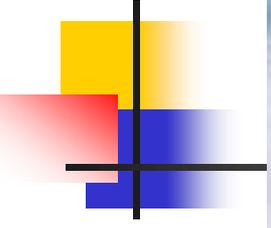
Blending to an edge



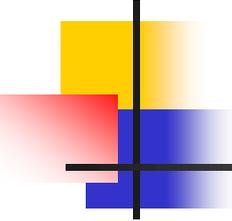
Union

Min/max

R-function

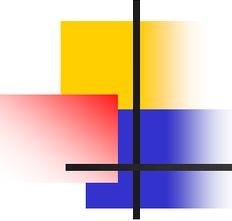


Blending Union

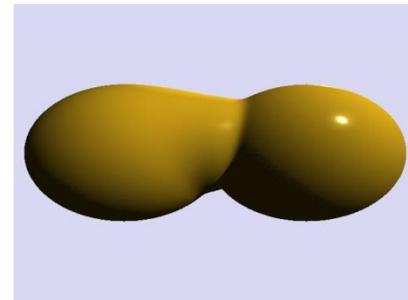
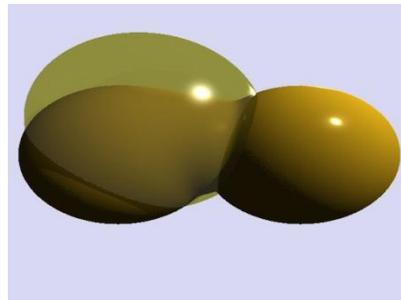
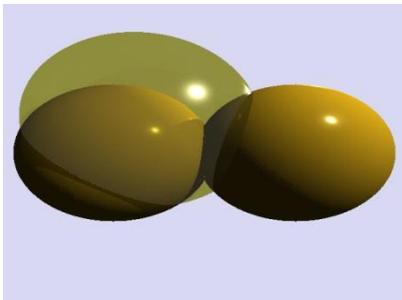
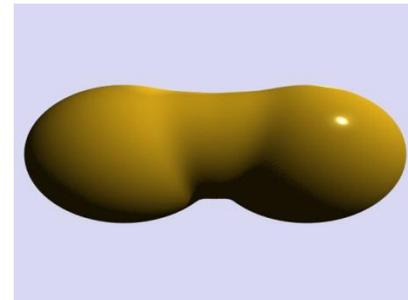
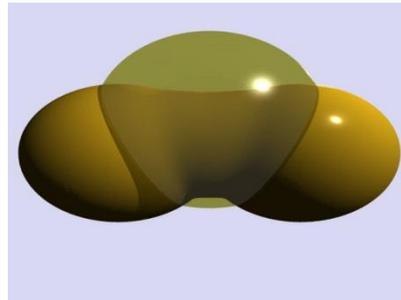
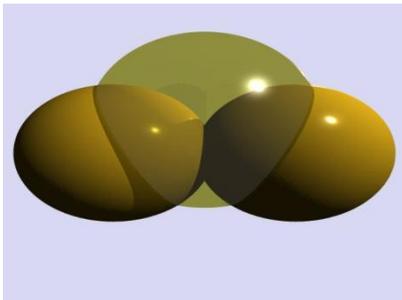
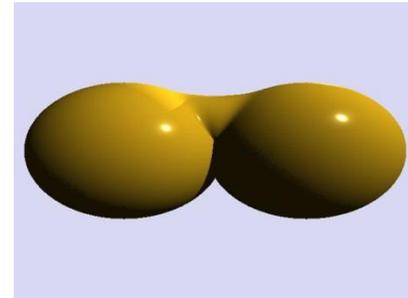
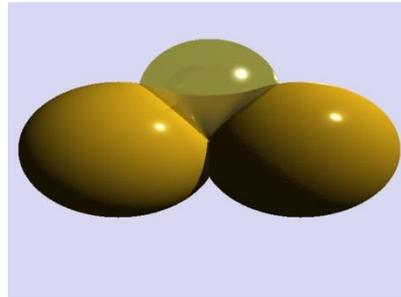
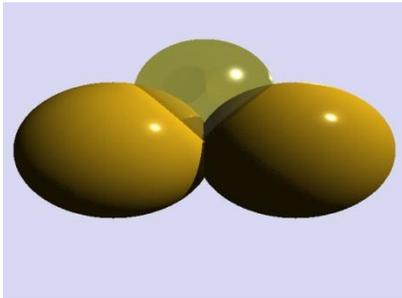


Bounded displacement function

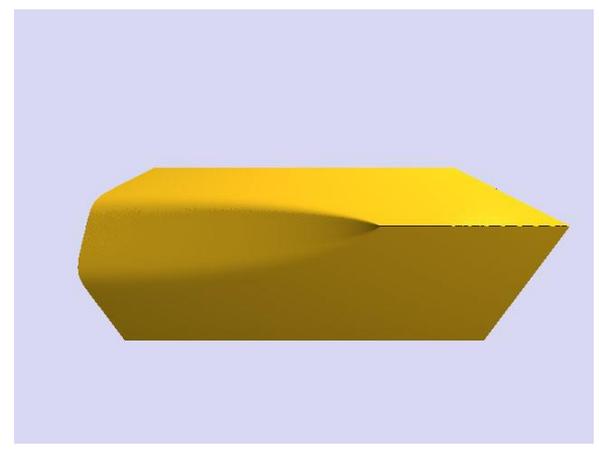
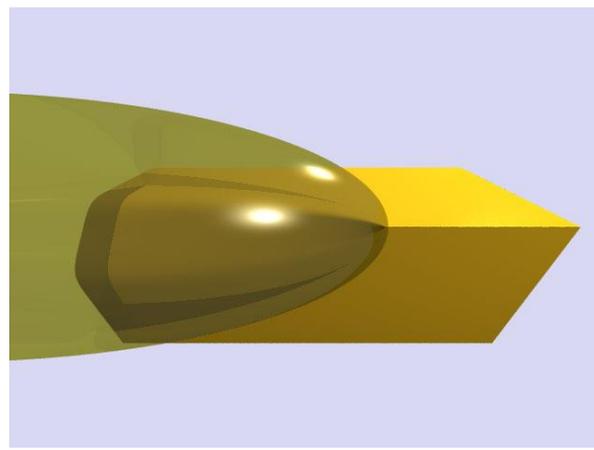
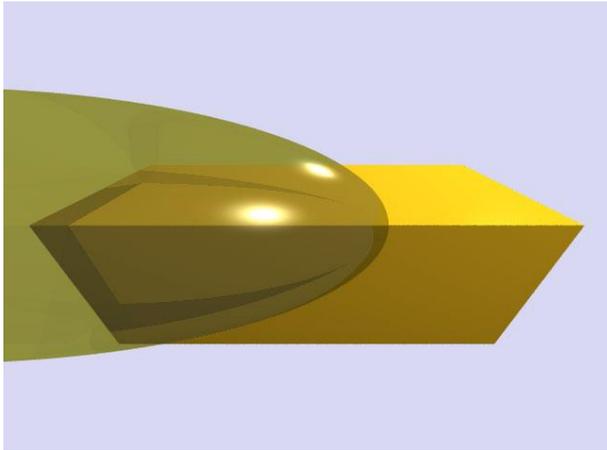
- 1) $disp_{bb}(r)$ takes the maximal value for $r=0$;
- 2) $disp_{bb}(r) = 0, r \geq 1$
- 3) $\frac{\partial disp_{bb}}{\partial r} = 0, r = 1$ the curve tangentially approaches the axis at $r=1$.



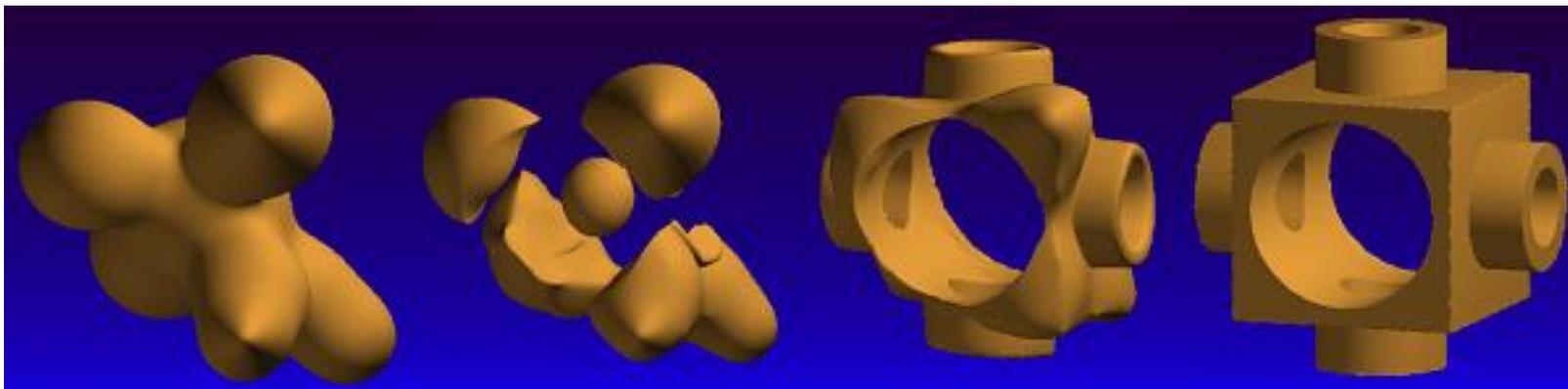
Bounded Blending



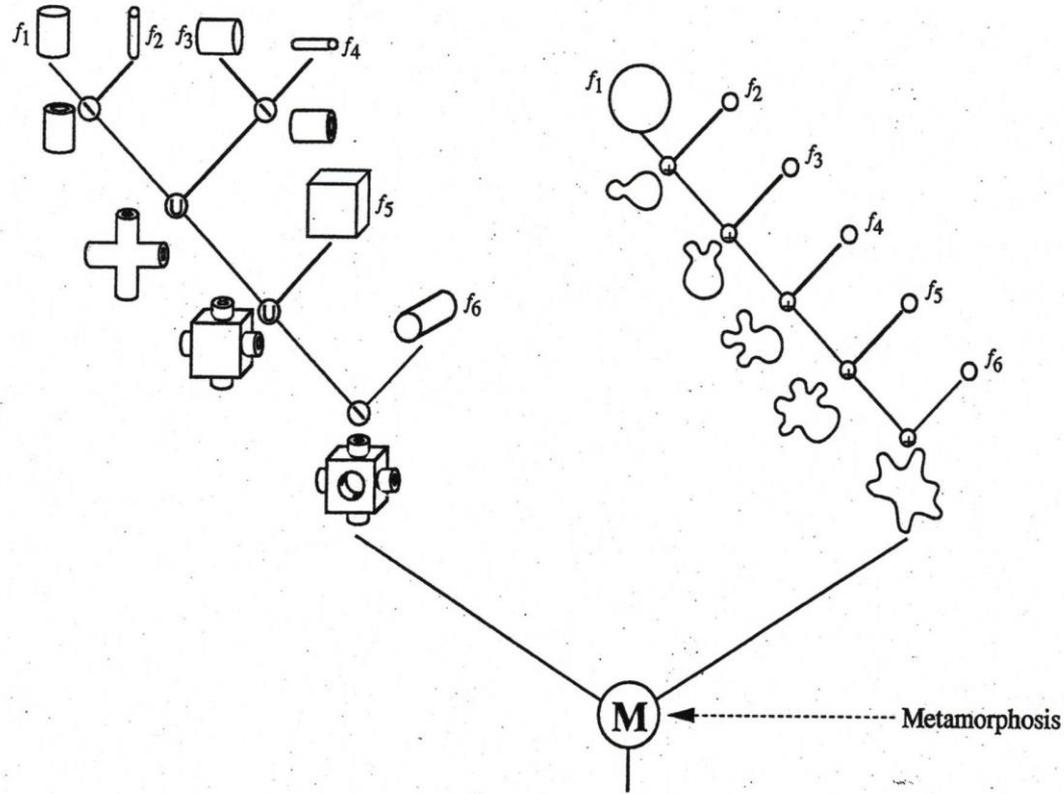
Partial edge blending



Metamorphosis

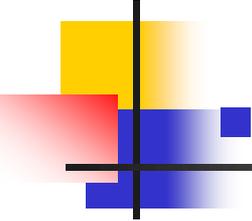


Construction tree



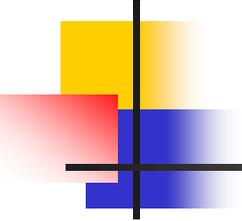
$$M(t) = F_1 \cdot t + F_2(1 + t)$$

What is FRep, anyway?

- 
- Uniform representation of multidimensional solids defined as

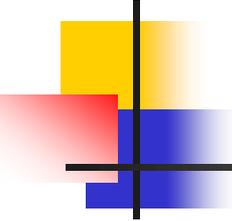
$$F(X) \geq 0$$

- Function $F(X)$ evaluation procedure traversing the construction tree structure
- Leaves: primitives
- Nodes: operations + relations
- “Empty Case” principle and extensibility



FRep Advantages

- Compact symbolic models
- Compression > 1000 times
- Web and mobile apps
- Precise models – infinite resolution
- Arbitrary parameterization
- Procedurally generated
- Uniform model for geometry and heterogeneous volume materials



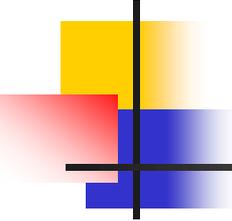
Software Tools

Specific forms:

- SvLis [Bowyer et al.]
- BlobTree [B. Wyvill et al.]
- MeshUp [Uformia]

Full FRep:

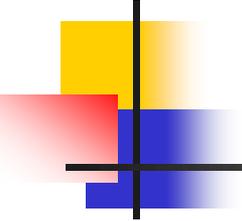
- HyperFun [Adzhiev et al.]
- Symvol for Rhino [Uformia]

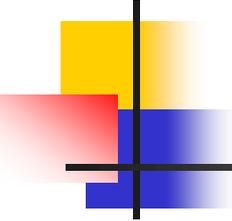


General References

- Pasko A., Adzhiev V., Sourin A., Savchenko V., **Function representation in geometric modeling: concepts, implementation and applications**, The Visual Computer, vol. 11, No. 8, 1995, pp. 429-446.

General References

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- Rvachev V.L., **On the analytical description of some geometric objects**, Reports of Ukrainian Academy of Sciences, vol. 153, No. 4, 1963, pp. 765-767 (in Russian).
 - Ricci A., **A constructive geometry for computer graphics**, The Computer Journal, vol. 16, No. 2, 1973, pp. 157-160.



More References

- Savchenko V., Pasko A., **Transformation of functionally defined shapes by extended space mappings**, The Visual Computer, vol. 14, No. 5/6, 1998, pp. 257-270.
- Sourin A., Pasko A., **Function representation for sweeping by a moving solid**, IEEE Transactions on Visualization and Computer Graphics, vol. 2, No. 1, 1996, pp.11-18.